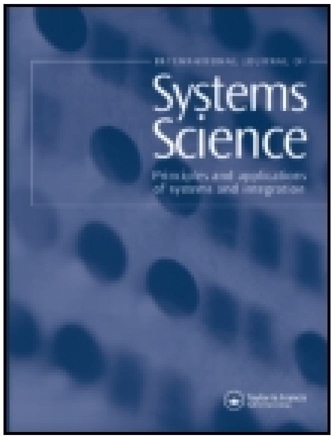

International Journal of Systems Science
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tsys20

Supply-chain modelling and control under proportional inventory-replenishment policies

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Published online: 19 May 2008.

International Journal of Systems Science
Vol. 39, No. 7, July 2008, 699–711

Supply-chain modelling and control under proportional inventory-replenishment policies

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(*Received 10 December 2007; final version received 3 February 2008*)

A novel state-space model of a multi-node supply chain is presented, controlled via local proportional inventory-replenishment policies. The model is driven by a stochastic sequence representing customer demand. The model is analysed under stationarity conditions and a simple recursive scheme is developed for updating its covariance matrix. This allows us to characterise the 'bullwhip effect' (demand amplification) in the chain and to solve an optimisation problem for a three-node model involving the minimisation of inventory subject to a probabilistic constraint on downstream demand. Finally, issues related to estimation schemes based on local historical data are briefly discussed.

Keywords: bullwhip effect; state-space model; supply chain; covariance matrix estimation; information-sharing

1. Introduction

The work presented in this article aims to analyse the effects of certain aspects of proportional (continuous) inventory policies on the stability and performance of serial multi-node supply chains. A short version of the article has appeared in Papanagnou and Halikias (2006).

In contrast to more traditional inventory-replenishment policies commonly used for supply chain control (e.g. (S, s) policies), continuous policies (e.g. P or PI policies) have only recently been proposed, apparently inspired from the area of classical process control engineering (Perea-Lopez, Grossmann, and Ydstie 2001; Dejonckheere, Disney Lambrecht, and Towhill 2003; Lin, Wong, Jang, Shieh, and Chu 2004). Their main characteristic is that orders take place continuously, rather than being triggered by specific events (e.g. when the inventory falls below a certain target level). Despite possible practical limitations of continuous ordering policies in some cases, these in principle can offer additional flexibility (e.g. by smoothing out flows), which can be beneficial for the stability and performance properties of the supply chain. In practise, continuous ordering policies are applicable when cost savings due to batch ordering are not significant.

Relative stability in supply chain dynamics is often quantified via the concept of 'bullwhip effect'. The bullwhip effect is a well-known instability phenomenon in supply chains, related to increased volatility in demand profiles in the upstream nodes of the chain (Forrester 1961; Sterman 1989). This may limit significantly the smooth operation of the chain and result in high costs arising due to its implications on production planning, high levels of inventory costs, poor customer service, etc. (Simchi-Levi et al. 2003). The bullwhip effect has been analysed extensively in recent literature, and many contributing factors for this phenomenon have been identified (Lee, Padmanabhan, and Whang 1997; Pera-Lopez et al. 2001; Dejonckheere et al. 2003). These include poor coordination, aggressive stock replenishment/demand forecasting policies and uncertain lead times in the chain. Note that these factors apply for general ordering policies, not only proportional policies considered in this article. In this work, we will present explicit methods for analysing and predicting the bullwhip effect via covariance analysis of proportional control schemes in supply-chain models of arbitrary complexity. Moreover, we study issues related to supply-chain performance under such schemes, the potential advantages of information-sharing and the applicability of local estimation schemes based on historical data. The benefits of using a state-space (rather than a transfer-function) approach arise mainly from its suitability for the recursive updating of the covariance matrix of structured multi-node systems of the type used in this work. Moreover, the covariance analysis undertaken in the first part of the work provides important information on the overall stability and performance of the chain, which is not directly available by other means of analysis.

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ISSN 0020–7721 print/ISSN 1464–5319 online

DOI: 10.1080/00207720802090898
http://www.informaworld.com

An additional feature of our article is that the model is driven by a stochastic process representing customer demand, which is initially assumed to be 'white', i.e. a time-series of uncorrelated normally distributed random variables. In case this is not a realistic representation of customer demand profiles, we can always use a filtered version of this signal via an ARMA model (Zipkin 2000; Aviv 2003) to generate arbitrary spectral characteristics representing more complex correlation patterns, seasonal variations, etc. A specific illustration involving the analysis of the bullwhip effect for a three-node chain with a first-order AR filter in provided in a later section of the article.

The main objective of the article is to derive results based on a generic supply-chain model, which is easy to analyse quantitatively but at the same time is sufficiently generic to capture the essential issues under investigation, which includes: (a) The analysis of the bullwhip effect in serial multi-node chains, arising especially due to aggressive ordering policies; (b) Issues of optimisation under information-sharing and their effect on the overall stability and performance of the chain (e.g. customer satisfaction levels); and (c) The possibility of estimation of policy parameters of adjacent nodes using only local historical data. Thus, we do not consider explicitly multiple vendors on either the upstream or downstream side of a particular node and the flow of orders and products through a specific node is interpreted in aggregate terms (i.e. as arising from multiple sources).

2. The supply-chain model

A simple series multistage supply-chain is considered as shown in Figure 1. There are n individual stages between generic Customer and Manufacturer and we denote as i the intermediate supplier index ($i \geq 1$). Figure 1 also depicts the flow of goods and information (orders) within the supply-chain. Let $I_i(t)$ denote the inventory level of node i at time t. We let also $Y_{i,i-1}(t)$ indicate the amount of goods to be delivered to node $i-1$ by the upstream node i at time instant t. We also introduce a time delay L, which is the lead time needed for the goods to be dispatched to the downstream node (i.e. the goods dispatched at time t are delivered at time $t+L$). For further analysis, we assume that $L=1$. The model is based on Lin et al. (2004), from where additional details can be obtained, including the main linearising assumptions used to make the mathematical analysis tractable. We consider the supply-chain network as a decentralised control system where there is no global moderator and decisions are taken locally at each node.

Balancing the inventory $I_i(t)$ of node i at time step t gives:

$$I_i(t) = I_i(t-1) + Y_{i+1,i}(t-L) - Y_{i,i-1}(t) \qquad (1)$$

where $I_i(t-1)$ is the inventory level at node i at time step $t-1$ and $Y_{i,i+1}(t-L)$ represents the products dispatched by the upstream node $i+1$ to node i, which are assumed to arrive with a delay of L time steps. Although inventory level is a key variable in supply-chain operation, each node i can better monitor the changes in inventory level at time t by using inventory position, $IP_i(t)$, which is given by:

$$IP_i(t) = IP_i(t-1) + Y_{i+1,i}(t) - Y_{i,i-1}(t) \qquad (2)$$

We denote by $O_{i,i+1}(t)$ the amount of orders placed by node i to node $i+1$, given by:

$$O_{i,i+1}(t) = k_i(SP_i - IP_i(t)) \qquad (3)$$

where SP_i represents a target set-point (assumed constant) and k_i is the corresponding inventory replenishment gain factor.

For the purposes of further analysis, it is assumed that $Y_{i,i-1}(t) = O_{i-1,i}(t-1)$. This implies that the amount of goods dispatched from node i to the downstream node $i-1$ at time t is the amount of orders placed on node i at time $t-1$. This is essentially a linearisation assumption as it assumes that there is always enough inventory to meet downstream demand. This assumption is also made in Lin et al. (2004) as it simplifies the subsequent analysis.

The above equations for the ith node may be written more compactly in state-space form by selecting $IP_i(t-1)$ and $Y_{i,i-1}(t)$ as state-space variables. The input and output variables of the ith node are also selected as $(O_{i-1,i}(t), Y_{i+1,i}(t))$ and $(Y_{i,i-1}(t), O_{i,i+1}(t))$, respectively.

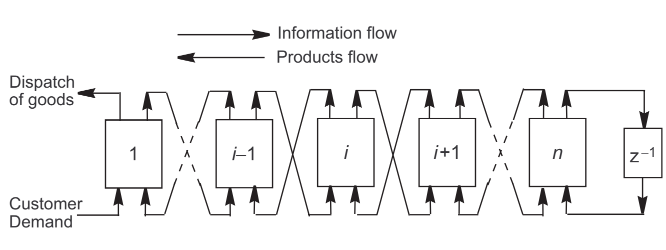

Figure 1. Series supply chain with n stages.

With this choice, the state-space model of the ith node can be written more compactly as:

$$\begin{pmatrix} IP_i(t) \\ Y_{i,i-1}(t+1) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} IP_i(t-1) \\ Y_{i,i-1}(t) \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} O_{i-1,i}(t) \\ Y_{i+1,i}(t) \end{pmatrix}$$

and

$$\begin{pmatrix} Y_{i,i-1}(t) \\ O_{i,i+1}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k_i & k_i \end{pmatrix}\begin{pmatrix} IP_i(t-1) \\ Y_{i,i-1}(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -k_i \end{pmatrix}\begin{pmatrix} O_{i-1,i}(t) \\ Y_{i,i+1}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ k_i \end{pmatrix} SP_i$$

The equivalent state-space model of the manufacturer (node $n+1$) is:

$$x_\phi(t+1) = A_\phi x_\phi(t) + B_\phi O_{n,n+1}(t)$$
$$\text{and} \quad Y_{n+1,n}(t) = C_\phi x_\phi(t)$$

where x_ϕ denotes the state of node $n+1$. We shall assume that the manufacturer acts as a pure time-delay, i.e. he is able to meet the orders placed on him with a delay of one time step. Consequently, we simply have that $Y_{n+1,n}(t) = O_{n,n+1}(t-1)$ and we can select $A_\phi = 0$ and $B_\phi = C_\phi = 1$.

A state-space realisation of the whole chain $(n+1)$ can be obtained by augmenting the realisations of all $n+1$ nodes. The full derivation is included in Appendix 1. As an example, the state-space realisation of the three-node chain is given as:

$$\begin{pmatrix} IP_1(t) \\ Y_{1,0}(t+1) \\ IP_2(t) \\ Y_{2,1}(t+1) \\ Y_{3,2}(t+1) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ -k_1 & k_1 & 0 & -k_1 & 0 \\ 0 & 0 & -k_2 & k_2 & -k_2 \end{pmatrix} \times \begin{pmatrix} IP_1(t-1) \\ Y_{1,0}(t) \\ IP_2(t-1) \\ Y_{2,1}(t) \\ Y_{3,2}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} O_{0,1}(t) + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ k_1 & 0 \\ 0 & k_2 \end{pmatrix}\begin{pmatrix} SP_1 \\ SP_2 \end{pmatrix}$$

which is of the form $x(t+1) = Ax(t) + Be(t) + F(SP)$. Note that for an $(m+1)$th-node mode (including the manufacturer's terminal node) the dimension of the A, B and F matrices are $(2m+1) \times (2m+1)$, $(2m+1) \times 1$ and $(2m+1) \times m$, respectively. We make this dependence explicit in the following section (where models with various number of nodes are considered) by writing the state-transition matrix as $A = A_{2m+1}$.

3. Computation of model's covariance matrix

In this section, we outline a method for calculating the covariance matrix of the state-vector $x(t)$ of the overall model developed in the previous section using symbolic computations. In our application, symbolic computations are essential, since we wish to obtain the solution as a function of the gain parameters $\{k_i\}$, which will allow further investigation of the bullwhip effect using our model. We first outline a general solution method based on Kronecker matrix products and vectorisation operations (Horn and Johnson 1994) subsequently, the special structure of the state-space model is exploited to derive a simple recursive solution procedure which can be applied to models of arbitrarily high complexity. Proofs for all results of this section can be found in Papanagnou and Halikias (2005).

Consider the LTI discrete-time state-space model: $x(t+1) = Ax(t) + Be(t)$, $y(t) = Cx(t)$, where $\{e(t)\}$ denotes a white vector-noise sequence of unit intensity, representing customer demand, assumed to have been applied as the input to the model since the infinite past. Then, assuming that A is asymptotically stable (all eigenvalues of A have modulus <1), the (steady-state) covariance of the state-vector $x(t)$, $E[x(t)x'(t)]$, is given by the (unique, positive semi-definite) solution of the discrete Lyapunov equation: $P - APA' - BB' = 0$. Further (Davies and Vinter 1985), $E(yy') = CPC'$. In our case, A depends linearly on n parameters $k_1, k_2, \ldots, k_n$, which are assumed constant (but possibly unknown). Hence, the solution of the above Lyapunov equation is the steady-state covariance of $x(t)$ for all combinations of $\{k_i\}$ for which A is asymptotically stable. It is shown next that this condition is satisfied if and only if the parameter vector $k = (k_1, k_2, \ldots, k_n)$ lies in the hypercube:

$$\mathcal{K}_n = (0,2)^n := \{k \in \mathcal{R}^n : 0 < k_i < 2, i = 1, 2, \ldots, n\}$$

This agrees with a parallel result in Dejonckhecen et al (2003).

Lemma 1: *Consider the $(m+1)$th-node model depending on m real gain parameters $k = \{k_1, k_2, \ldots, k_m\}$. Then the system is asymptotically stable if and only if $k \in \{\mathcal{K}\}_m$. In particular, if $A = A_{2m+1}$ denotes the 'A'-matrix of the state-space realisation of the system, then the eigenvalues of A are $\{1-k_1, 1-k_2, \ldots, 1-k_m, 0, \ldots, 0\}$, where the multiplicity of the zero eigenvalue is $m+1$.*

Next, let $A \otimes B$ denote the Kronecker product of two matrices A and B; let also vec(A) be the operation that stacks the elements of a matrix A in a column vector (sweeping along the rows of A). Applying the vec($\cdot$) operation to the Lyapunov equation gives $(I_{n^2} - A \otimes A)\mathrm{vec}(P) = \mathrm{vec}(BB')$ where $n = 2m+1$ (see, Horn and Johnson (1994)), which may be solved as: $\mathrm{vec}(P) = (I_{n^2} - A \otimes A)^{-1}\mathrm{vec}(BB')$. The next, Lemma 2 guarantees that the inverse exist sabove indicated.

Lemma 2: *Matrix $I_{n^2} - A \otimes A$ is non-singular for all k such that $k \in \mathcal{K}_m$. In fact, $I_{n^2} - A \otimes A$ is singular if and only if $(1-k_i)(1-k_j) = 1$ for any two indices i and j such that $1 \le i \le m$ and $1 \le j \le m$, where $n = 2m+1$.*

The calculation of the covariance matrix P essentially involves the solution of a system of n^2 linear equations in the elements of P, which depend parametrically on the k_i's. Since the solution of the Lyapunov equation is symmetric, however, this system of equations is redundant (with $n(n-1)/2$ equations being repeated (Horn and Johnson 1990)). The solution can be simplified using the following procedure: for a symmetric matrix P, let $\overline{\mathrm{vec}}(P)$ denote vec(P) with all the entries of P below the main diagonal eliminated. Clearly, if $P \in \mathcal{R}^{n\times n}$, then $\overline{\mathrm{vec}}(P) \in \mathcal{R}^r$, where $r = n(n+1)/2$. Define $W \in \mathcal{R}^{n^2 \times r}$ so that $\mathrm{vec}(P) = W\overline{\mathrm{vec}}(P)$. Let also $\mathcal{S} \subset \{1, 2, \ldots, n\}$ be the subset of the $n(n-1)/2$ indices of vec(P), which are eliminated when constructing $\overline{\mathrm{vec}}(P)$. Then we can write $V(I_{n^2} - A \otimes A)W\overline{\mathrm{vec}}(P) = V\mathrm{vec}(BB')$ where $V \in \mathcal{R}^{r\times n^2}$ denotes the unit matrix with all rows corresponding to indices in $\mathcal{S}$ eliminated. Clearly, multiplication from the right by matrix V eliminates the $n(n-1)/2$ redundant equations. Further we have:

Lemma 3: *Matrix $V(I_{n^2} - A \otimes A)W$ is non-singular for all $k \in \mathcal{K}_m$.*

Using Lemma 3, we can obtain the unique solution $p = \overline{\mathrm{vec}}(P) = [V(I_{n^2} - A \otimes A)W]^{-1}V\mathrm{vec}(BB')$ from which P can be recovered as $P = \overline{\mathrm{vec}}^{-1}(p)$.

Example: Using the two methods described in the earlier part of this section, the covariance matrices corresponding to the three-node models were obtained using the symbolic Matlab toolbox, as:

$$P_5 = \begin{pmatrix} \frac{1}{k_1(2-k_1)} & 0 & -\frac{k_1-1}{(k_1-2)k} & \frac{1}{k_1-2} & \frac{(k_1-1)k_2}{(k_1-2)k} \\ 0 & 1 & 0 & 0 & 0 \\ -\frac{k_1-1}{(k_1-2)k} & 0 & \frac{k_1(k+2)}{k_2(2-k_1)(k_2-1)k} & \frac{(k_1-1)k_1}{(k_1-2)k} & \frac{(k+2)k_1}{(k_1-2)(k_2-2)k} \\ \frac{1}{k_1-2} & 0 & \frac{(k_1-1)k_1}{(k_1-2)k} & \frac{k_1}{2-k_1} & -\frac{(k_1-1)k_1k_2}{(k_1-2)k} \\ \frac{(k_1-1)k_2}{(k_1-2)k} & 0 & \frac{(k+2)k_1}{(k_1-2)(k_2-2)k} & \frac{(k_1-1)k_1k_2}{(2-k_1)k} & \frac{(k+2)k_1k_2}{(k_1-2)(2-k_2)k} \end{pmatrix} \tag{4}$$

where $k = k_1k_2 - k_2 - k_1$.

A superior method for calculating the covariance matrix of the state-vector is to use the special structure of the state-space model, which leads to a simple recursive updating algorithm. This is outlined in the following result:

Lemma 4: *Let (A_{2j+1}, B_{2j+1}) denote the $(j+1)$th node state-space model, depending on the j parameters $\{k_1, k_2, \ldots, k_j$ where $j \ge 1$. Then:*

(1) *There is a state-space transformation defined by a permutation matrix Q_j, such that*

$$Q_jA_{2j+1}Q_j := A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$$

and $Q_jB_{2j+1} = B_{2j+1} := B$ in which: (i) $A_{11} = A_{2j-1}$, (ii) *A_{21} and A_{22} have rank one, and* (iii) *B is of the form $[B_1' \; 0_{2j-1}]'$.*

(2) *The Lyapunov equation $P - APA' - BB' = 0$ has a unique symmetric positive-semidefinite solution P for all $(k_1, k_2, \ldots, k_j) \in (0, 2)^j$. Let P be partitioned conformally with A, i.e,*

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22} \end{pmatrix}$$

where $P_{11} = P_{11}' \in \mathcal{R}^{(2j-1)\times(2j-1)}$, $P_{12} \in \mathcal{R}^{(2j-1)\times 2}$ and $P_{22} = P_{22}' \in \mathcal{R}^{2\times 2}$. Then $P_{11} = P_{2j-1}$ where P_{2j-1} is the covariance matrix of the j th-node model, i.e, the unique symmetric solution of the discrete Lyapunov equation: $P_{2j-1} - A_{2j-1}P_{2j-1}A_{2j-1}' - B_{2j-1}B_{2j-1}' = 0$. Further, P_{12} and P_{22} have rank at most one and may be obtained from the unique solutions of the linear equations: $P_{12} - A_{11}P_{12}A_{22}' = A_{11}P_{11}A_{21}'$ and $P_{22} - A_{22}P_{22}A_{22}' = A_{21}P_{11}A_{21}' + A_{22}P_{12}' A_{21}' + A_{21}P_{12}A_{22}'$, respectively.

(3) *If $(k_1, k_2, \ldots, k_j) \in (0, 2)^j$, the Lyapunov equation: $P_{2j+1} - A_{2j+1}P_{2j+1}A_{2j+1}' - B_{2j+1}B_{2j+1}' = 0$ has a unique symmetric positive semi-definite solution given by:*

$$P_{2j+1} = Q_j\begin{pmatrix} P_{2j-1} & P_{12} \\ P_{12}' & P_{22} \end{pmatrix}Q_j$$

Remark: The Lemma 4 shows that the covariance matrix of the $j+1$th-node model may be obtained recursively from the solution of the jth-node model by solving two linear equations of order $2(2j-1)$ and 4, respectively (in fact of order $2j-1$ and 2, taking into account that P_{12} and P_{22} both have rank of at most one). This can be achieved by the vectorisation approach outlined earlier. Hence, the bulk of the computation involving the solution of a $(2j-1)\times(2j-1)$ matrix equation is completely avoided. After P has been assembled from P_{2j-1}, P_{12} and P_{22}, P_{2j+1} may be obtained by reversing the permutation through matrix Q_j.

4. Characterisation of bullwhip effect

The covariance analysis of the model allows us to analyse the effect of the inventory-replenishment policies on the bullwhip effect. Recall that end-customer demand $O_{0,1}(t)$ has been modelled as a sequence of independent and identically distributed random variables of unit variance. Hence, the variance of the demand signal at any node of the chain may be calculated easily from the covariance matrix. Consider the three-node model. The orders placed by the second node (on the manufacturer) correspond to signal $O_{2,3}(t)$ and we can write:

$$\begin{aligned} O_{2,3}(t) &= Y_{3,2}(t+1) \\ &= -k_2 IP_2(t-1) + k_2 Y_{2,1}(t) - k_2 Y_{3,2}(t) + k_2 SP_2, \end{aligned}$$

which can be written as a linear combination of the state-variables (and SP_2) in the form $O_{2,3}(t) = C'x(t) + k_2 SP_2$ where $x(t)$ is the state-vector of the model and $C' = (0\ 0\ -k_2\ k_2\ -k_2)$. The bullwhip effect, representing the amplification in order 'fluctuations' placed on nodes 1 and 3 is given by:

$$\begin{aligned} \beta &= \frac{\mathrm{Var}(O_{2,3})}{\mathrm{Var}(O_{0,1})} = C'P_5C \\ &= \frac{k_1k_2(2+k_1k_2-k_1-k_2)}{(2-k_1)(2-k_2)(k_1+k_2-k_1k_2)} \end{aligned} \tag{5}$$

To find the regions in the (k_1, k_2) plane where demand amplification and demand attenuation occurs, β was set to one, and the resulting equation was solved to give k_2 as a function of k_1. This gives two solutions:

$$k_2 = f(k_1) = \frac{2-5k_1+2k_1^2 \pm \sqrt{4-12k_1+13k_1^2-4k_1^3}}{2(k_1-1)^2}, \tag{6}$$

which are valid for $k_1 \neq 1$. It can be easily seen that the positive square root should be chosen, as with this choice, k_1 values in the interval $0 \le k_1 \le 2$ are mapped to k_2 values inside the same interval. The resulting curve is plotted in Figure 2 (for $\alpha = 0$), and indicates the boundary between the demand-amplification and demand-attenuation regions. As expected, aggressive-replenishment policies (i.e. large values of k_1 and k_2) reinforce the bullwhip effect. The analysis can be easily extended for correlated profile signals, modelled via arbitrary ARMA models driven by white noise of (unit) intensity $e(t)$. Assuming, for example, that $O_{0,1}$ is a first-order autoregressive (AR), i.e.

$$O_{0,1}(t+1) = \alpha O_{0,1}(t) + (1-\alpha)e(t)$$

where α is a correlation (smoothing) parameter lying in the interval $-1<\alpha<1$, it can be shown that the demand-attenuation factor region expands at the expense of the demand-amplification region as shown in Figure 2, plotted for values $\alpha = 0$, 0.2, 0.5 and 0.8. Note that, as expected, 'smoother' customer demand fluctuations result in the alleviation of the bullwhip effect.

5. Optimal policies under information-sharing

In this section we specialise our system to a three-node model. We still assume linear dynamics (i.e. that all inventories are sufficiently high to meet downstream demand with no back-orders). The manufacturer (node 3) is again modelled as a unit delay, i.e. he delivers the requested products with

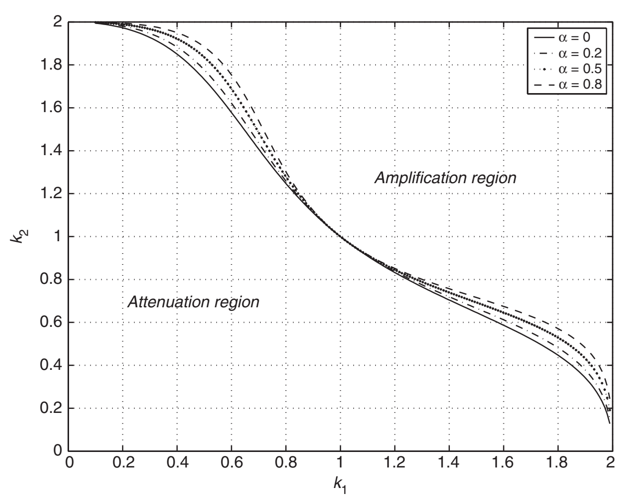

Figure 2. Boundary between demand-amplification and -attenuation regions as a function of α.

a delay of one time period. Assume further that customer demand is normally distributed as $e(t)=O_{0,1}(t)\sim N(\mu,\sigma^2)$. Provided that the system is stable ($0<k_1<2$ and $0<k_2<2$), all signals in the limit are stationary.

Remark: The assumption that customer demand is normally distributed is not essential for the analysis of this section and can be easily removed (by assuming that the customer demand signal is made up of independent and identically distributed random variables). This is also true of the local estimation schemes presented in the next section, which do not rely on *a priori* knowledge of any specific distribution. The normality assumption is only made for the purposes of obtaining distributions for the various variables of the model and for supporting the main results via concrete simulations.

The expected values of the state-variables can be found using the state-space model, which is of the form:

$$x(t+1)=Ax(t)+Be(t)+F(SP)$$

where SP is the (deterministic) vector of set-points $SP=(SP_1\ \ SP_2)'$ (assumed constant). Thus, under stationary conditions,

$$x(t)=(I-A)^{-1}Be(t)+(I-A)^{-1}F(SP)$$

and hence,

$$E[x(t)]=\mu(I-A)^{-1}B+(I-A)^{-1}F(SP)$$

Note that the indicated matrix inverse exists as the spectral radius of $A<1$ as long as $0<k_1<2$ and $0<k_2<2$. Thus, the five state-variables are distributed as:

$$IP_1(t)\sim N\left(SP_1-\frac{\mu}{k_1},\frac{\sigma^2k_1}{2-k_1}\right)$$

$$Y_{1,0}(t)\sim N(\mu,\sigma^2)$$

$$IP_2(t)\sim N\left(SP_2-\frac{\mu}{k_2},\frac{\sigma^2k_1(k_1k_2-k_1-k_2+2)}{k_2(2-k_1)(2-k_2)(k_1+k_2-k_1k_2)}\right)$$

$$Y_{2,1}(t)\sim N\left(\mu,\frac{\sigma^2k_1}{2-k_1}\right)$$

$$O_{2,3}(t)\sim N\left(\mu,\frac{\sigma^2k_1k_2(k_1k_2-k_1-k_2+2)}{(2-k_1)(2-k_2)(k_1+k_2-k_1k_2)}\right)$$

Next, we define a new variable (excess inventory position), $EI_2(t)=IP_2(t-1)-O_{1,2}(t)$, which monitors the ability of node 2 to meet the downstream demand placed on it. It follows that:

$$EI_2(t)\sim N\left(SP_2-\frac{\mu(k_2+1)}{k_2},\sigma^2_{EI_2}(k_1,k_2)\right)$$

where

$$\sigma^2_{EI_2}(k_1,k_2)=\frac{\sigma^2k_1(2-5k_2k_1+k_2^2k_1-k_2^3+k_2^3k_1-k_1+3k_2+4k_2k_1^2-2k_2^2k_1^2)}{k_2(2-k_2)(k_1+k_2-k_2k_1)(2-k_1)}$$

Note that under step demand, IP_i ($i=1,2$) does not track the set-point SP_i exactly, but with a steady-state error μ/k_i, which is a characteristic of type-zero feedback systems. As expected, the information pattern is asymmetric, i.e. node 2 (Distributor) is affected by the inventory policy of node 1 (Retailer) but not vice versa. Suppose now that the Retailer makes his policy gain-factor k_1 known. In this case, the Distributor can make use of this information to minimise his own costs, typically related to excessive inventory levels. Although this objective is situation-specific (e.g. due to possible existence of capacity constraints, depreciation effects, etc), it is reasonable to assume that the objective of the Distributor is to minimise both his average inventory and his inventory fluctuations. Note that in our model the Distributor is always capable of controlling his average inventory level through his choice of SP_2, which can be used to shift $E(IP_2)$ to any required level.

An additional requirement is that the Distributor should have enough inventory to meet (fluctuating) downstream demand, at least for most orders placed on him. This is in order to ensure the smooth operation of the chain, to which he has an interest as a participant. One way of modelling this requirement is to include explicit penalty-terms in the Distributor's 'objective function', reflecting real or virtual costs (e.g. penalty terms for not fulfilling a contract, loss of sales due to Customer dissatisfaction, etc). Here we impose a probabilistic constraint for fulfilling orders, i.e. we require that $\text{Prob}[EI_2<0]\leq\delta$ for some (small) parameter δ.

Let $\Phi(z)$ denote the cumulative distribution function of the normal distribution $N(0,1)$, i.e.:

$$\Phi(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}e^{-\xi^2/2}\,d\xi$$

Then, using the distribution of $EI_2(t)$ above, the 'order-fulfilling' constraint takes the form:

$$SP_2-\frac{\mu(k_2+1)}{k_2}+\sigma_{EI_2}(k_1,k_2)\Phi^{-1}(\delta)\geq 0 \tag{7}$$

Thus, the optimisation problem faced by the Distributor is to choose his inventory replenishment policy parameters, k_2 and SP_2, to minimise his inventory costs subject to the constraint of Equation (7) (note that parameter k_1 is not under the control of the Distributor and has been assumed to be fixed and known).

Modelling inventory costs can be achieved in various ways depending on the specific practical situation faced by each Distributor and essentially involves the assessment of the relative importance attached to costs due to a high average inventory and to costs due to excessive inventory fluctuations. One of the following three approaches can be followed:

- The first approach, which is a compromise between the two trade-offs discussed above (i.e. mean *vs.* variance), is to assume that a price function $q(\xi)$ is attached to each possible inventory level. The function would be typically increasing and concave to reflect decreasing marginal costs. Then, the total expected inventory cost can be expressed as the weighted integral:

$$C(k_2, SP_2) = \frac{1}{\sqrt{2\pi}\sigma_{IP_2}} \int_0^\infty q(\xi)\exp(-(\xi-\mu_{IP_2})^2/2\sigma_{IP_2}^2)\mathrm{d}\xi$$

  assuming that $q(\xi)=0$ for $\xi \le 0$. Note that μ_{IP_2} and σ_{IP_2} (and thus, also the expected cost C) depend on the two parameters k_2 and SP_2 (assuming k_1 is fixed and known). An optimisation objective can then be formulated as the minimisation of $C(k_2, SP_2)$ subject to constraint (7) and solved via Lagrange multipliers. Although the problem is tractable in principle (at least for fixed values of the parameters), its solution is likely to depend critically on $q(\xi)$ and therefore would not reveal any interesting information about the optimal policies.
- The second approach places emphasis on the minimisation of the average inventory μ_{IP_2} (subject to constraint (7)). This method would be appropriate when the bulk of inventory costs is determined by the average inventory stored, rather than its fluctuations. To solve the problem in this case, note that $\mu_{EI_2}-\mu_{IP_2}=\mu$ (constant) and hence, constraint (7) can be written as $\mu_{IP_2} \ge \mu - \sigma_{EI_2}\Phi^{-1}(\delta)$. In the interesting case that δ is small (it suffices that $\delta \le 0.5$), $\Phi^{-1}(\delta) < 0$ and μ_{IP_2} is minimised by making the inequality (7) tight (i.e. equality) and minimising σ_{EI_2} (over k_2 for any fixed k_1). This would determine an optimal policy $k_2 = \hat{f}(k_1)$, say, from which the minimum value of μ_{IP_2} can be obtained as $\mu_{IP_2} = \mu - \sigma_{EI_2}\Phi^{-1}(\delta)$, where $k_2 = \hat{f}(k_1)$ has been substituted in the expression for σ_{EI_2}); note that the variance of IP_2 is also uniquely determined (as a function of k_1). Equation (7) (with equality) can then be used to determine the optimal setting of SP_2. We will not pursue this approach in detail.
- The third approach, which is analysed in detail, relates to the case when the bulk of inventory costs are due to excessive fluctuations in inventory stored, rather than the average inventory level. Here, for any given k_1 in the interval $0 < k_1 < 2$, we seek the optimal choice of k_2, $k_2^* = f^*(k_1)$ say, which minimises the variance of IP_2; subsequently we minimise the mean of IP_2 subject to constraint (7). Note that once the optimal policy $k_2^* = f^*(k_1)$ has been determined, we need to set:

$$SP_2^* = \frac{\mu(k_2^*+1)}{k_2^*} - \sigma_{EI_2}(k_1, k_2^*)\Phi^{-1}(\delta)$$

  resulting in the constrained minimum $\mu_{IP_2} = E[IP_2] = SP_2^* - (\mu/k_2^*)$. It has been assumed that the Customer-demand parameters μ and σ are known or can be estimated accurately from the data. The optimal solution in this case is summarised by the following proposition.

Proposition 1: *The minimising solution is given by*:

$$k_2 = f^*(k_1) = \frac{4\cos(\phi/3 - 2\pi/3) + 3k_1 - 4}{3(k_1-1)}$$

where $\phi = \phi(k_1)$ *is defined as*:

$$\phi = \tan^{-1}\left(\frac{3\sqrt{3k_1(2-k_1)(27k_1^2-54k_1+32)}}{27k_1^2-54k_1+16}\right) \tag{8}$$

Remark: Since $\tan^{-1}(\cdot)$ is a multifunction, it is stressed in Proposition 1 that ϕ take values in the interval $0 \le \phi < \pi$.

Proof: The proof follows via routine but complex manipulations by setting the derivative of the variance of IP_2 with respect to k_2 to zero and solving the resulting cubic equation to express k_1 as a function of $k_2 = f^*(k_1)$. Due to the high complexity of the expressions involved, Matlab's symbolic toolbox was used to verify the minimising solution in the interval $0 \le k_1 \le 2$. □

Note that $f^*(1)$ is not formally defined by the equation $k_2 = f^*(k_1)$, so we set $f^*(1) = 1$ to make the function continuous and continuously differentiable at $k_1 = 1$. A plot of $k_2^* = f^*(k_1)$ (along with the boundary between the attenuation and amplification regions) is shown in Figure 3. An important observation is that the optimal curve lies entirely in the attenuation region. Thus, under information-sharing (disclosure of policy parameter k_1 to the Distributor), a 'selfish' policy by the Distributor (resulting from his attempt to minimise his own inventory costs) cannot give rise to the bullwhip effect. Of course, this conclusion should be qualified by the assumptions of the model and the assumed form of the cost function.

The minimum variance Var*(IP_2) can be obtained by substituting the optimal policy $k_2^* = f^*(k_1)$ into the (3, 3) element of P_5. A plot of $\text{Var}^*(IP_2)$ *versus* k_1 reveals that $\text{Var}^*(IP_2)$ is a monotonically increasing function of k_1. The optimal curve $k_2^* = f^*(k_1)$ (which is a monotonically decreasing function) starts at point $(2, 0)$ (where $\text{Var}^*(IP_2) = 0$), passes through the point $(1, k_2^*(1) = 1)$ (where $\text{Var}^*(IP_2) = \sigma^2$) and approaches zero as $k_1 \to 2$ (where $\text{Var}^*(IP_2) \to \infty$ as this corresponds to the edge of the stability region).

Example: The results of the optimal policy $k_2^* = f^*(k_1)$ for three values of $k_1 = 0.5, 1, 1.5$ are summarised in Table 1. The parameters for the demand distribution were chosen as $\mu = 10$ and $\sigma = 1$, while parameter δ was set to $\delta = 0.05$. The distributions of IP_2 and $IP_2 - O_{1,2}$ for the three values of k_1 are shown in Figures 4 and 5 respectively. Note that all three distributions of $EI_2 = IP_2 - O_{1,2}$ suggest that inventory IP_2 is insufficient to meet downstream demand $O_{1,2}$ with probability 0.05, as set by parameter δ.

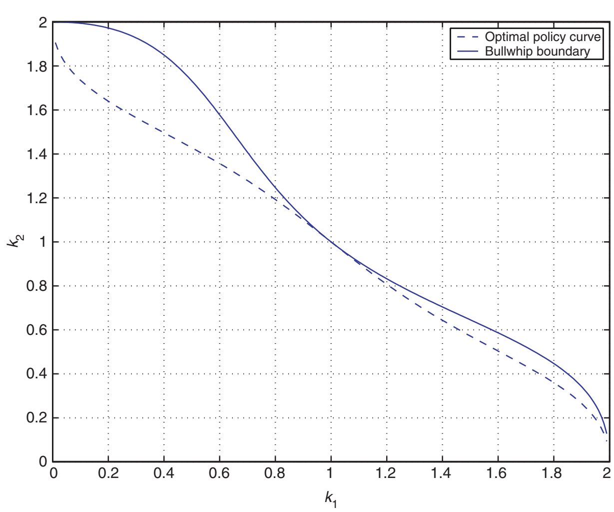

Figure 3. Optimal policy $k_2^* = f^*(k_1)$ and boundary between amplification and attenuation regions.

Table 1. Summary of optimal policy results.

k_1	k_2^*	E[IP_2]	Var[IP_2]	E[EI_2]	Var[EI_2]
0.50	1.43	11.41	0.26	1.41	0.73
1.00	1.00	12.33	1.00	2.33	2.00
1.50	0.57	14.23	2.38	4.23	6.61

6. Local estimation schemes

We continue our analysis of the three-node model by removing the assumption that policy parameter k_1 (corresponding to the Retailer's proportional replenishment policy) is communicated to the Distributor. A natural question arising in this case is whether k_1 can be estimated by the Distributor (node 2). Naturally, the data on which the estimation should be based are restricted only to the input/output and state variables local to node 2. In this section, we develop three estimation techniques based on the covariance matrix of the model. The first uses only partial information and can be

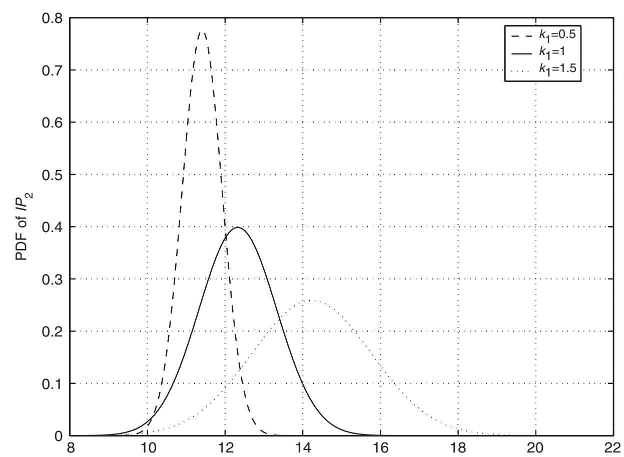

Figure 4. Probability density function of IP_2.

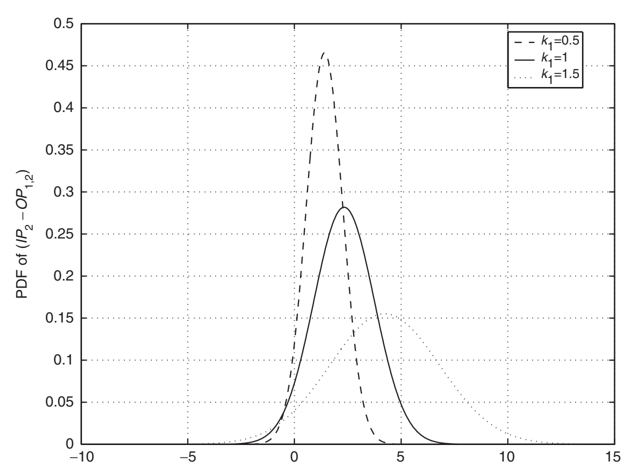

Figure 5. Probability density function of $EI_2 = IP_2 - O_{1,2}$.

implemented recursively. The second and third techniques use the full information of the part of the covariance matrix corresponding to the data available to the distributor. Note that in this section we do not make use of normality assumptions (remark in Section 5) and hence, the techniques are applicable to demand-profile data drawn from arbitrary distributions. We continue to assume, however, that these random variables are independent and identically distributed.

6.1. *Estimation method 1: partial information*

Since $E(Y_{2,1}) = \mu$, the mean customer demand (μ) can be estimated from $Y_{2,1}(t)$, which is an output signal of node 2 (e.g. an unbiased estimate $\hat{\mu}$ of μ can be obtained asymptotically). Consider next the part of the covariance matrix P_5 corresponding to the state-variables of node 2; this is the diagonal block of P_5 corresponding to the third and fourth rows and columns, i.e.:

$$\mathrm{Cov}(IP_2, Y_{2,1}) = \sigma^2 \begin{pmatrix} \dfrac{k_1(2-k_1-k_2+k_1k_2)}{k_2(2-k_1)(2-k_2)(k_1+k_2-k_1k_2)} & \dfrac{k_1(k_1-1)}{(2-k_1)(k_1+k_2-k_1k_2)} \\ \dfrac{k_1(k_1-1)}{(2-k_1)(k_1+k_2-k_1k_2)} & \dfrac{k_1}{2-k_1} \end{pmatrix} := (P_{ij}) \quad (9)$$

One way of estimating k_1 and σ is to define:

$$\alpha = \frac{P_{12}}{P_{22}} = \frac{k_1 - 1}{k_1 + k_2 - k_1 k_2} \Rightarrow k_1 = \frac{1 + \alpha k_2}{1 + \alpha k_2 - \alpha}$$

and note that:

$$\sigma^2 = \frac{P_{22}(2 - k_1)}{k_1}$$

Now, using the data $\{IP_2(t), Y_{2,1}(t)\}$ and noting that parameter k_2 is known, we can obtain estimates for $P_{11} = \mathrm{Var}(IP_2)$, $P_{22} = \mathrm{Var}(Y_{2,1})$ and $P_{12} = E[(IP_2 - E(IP_2))(Y_{21} - E(Y_{2,1})]$, say $\hat{P}_{11}, \hat{P}_{22}$ and $\hat{P}_{12}$, respectively, and use them to estimate k_1 and σ via equations:

$$\hat{\alpha} = \frac{\hat{P}_{12}}{\hat{P}_{22}}, \quad \hat{k}_1 = \frac{1 + \hat{\alpha} k_2}{1 + \hat{\alpha}(k_2 - 1)}, \quad \hat{\sigma}^2 = \frac{\hat{P}_{22}(2 - \hat{k}_1)}{\hat{k}_1}$$

This estimation scheme will produce asymptotically unbiased estimates for k_1 and σ^2 and can be implemented efficiently in a recursive fashion (Appendix 2).

6.2. *Estimation method 2: structured-covariance approximation*

A limitation of the first method is that it does not take full advantage of the available information structure (e.g. the information contained in $\mathrm{Var}(IP_2)$ is ignored). A superior approach is to formulate the estimation problem as a structured-covariance approximation, e.g.

$$\min_{k_1 \in (0,2), \sigma > 0} \left\| W \circ (\hat{P} - \sigma^2 P_5^{3,4}) \right\|_F^2$$

in which $\hat{P}$ denotes the estimated covariance matrix (constructed from the data) and $P_5^{3,4}$ denotes the submatrix of P_5 consisting of its third and fourth rows and columns (local information to node 2). The choice of Frobenious-norm makes the problem easily transformable into a scalar sum-of-squares type non-linear optimisation, while W is a weighting matrix, which can be used to emphasise/de-emphasise different matrix elements in the approximation (here '$\circ$' denotes the Hadamard product, i.e. element by element product, of two matrices (Horn and Johnson 1994). For example, choosing $W_{11} = W_{22} = 1$ and $W_{12} = W_{21} = (1/2)$ results in the objective function:

$$\begin{aligned} g(k_1) = {} & \left(\hat{P}_{11} - \frac{\sigma^2 k_1(2-k_1-k_2+k_1k_2)}{k_2(2-k_1)(2-k_2)(k_1+k_2-k_1k_2)}\right)^2 \\ & + \left(\hat{P}_{22} - \frac{\sigma^2 k_1}{2-k_1}\right)^2 \\ & + \left(\hat{P}_{12} - \frac{\sigma^2 k_1(k_1-1)}{(2-k_1)(k_1+k_2-k_1k_2)}\right)^2 \end{aligned}$$

which can be easily minimised (over $k_1 \in (0,2)$ and $\sigma^2 - 0$) via gridding or local search methods.

Example: We illustrate the estimation scheme by means of a simulation example. Assume that $O_{1,2} \sim N(\mu, \sigma^2)$ with $\mu = 10$ and $\sigma^2 = 1$. We simulate the three-node chain with parameters $k_1 = k_2 = 1.5$, $SP_1 = SP_2 = 20$ and $IP_1(0) = IP_2(0) = 20$ for $n = 1000$ time steps. Parameter k_1 is assumed unknown to node 2 (Distributor) and is estimated using the first method described earlier. The results of the estimation are summarised below in Table 2.

Applying the second estimation method described in this section (structured-covariance approximation) produced a (slightly) more accurate estimate $\hat{k}_1 = 1.51$. The minimisation was carried over k_1 using the estimated variance of the end-customer demand signal $\hat{\sigma}^2 = 1.09$. The graph of the cost function that

Table 2. Estimated and true parameters (method 1).

	$E[IP_1]$	$E[IP_2]$	$E[Y_{10}]$	$E[Y_{21}]$	$E[Y_{3,2}]$	$\mathrm{Var}[Y_{3,2}]$	α	σ^2	k_1
Estimated	13.35	13.35	9.99	9.99	9.98	15.94	0.63	1.09	1.48
True	13.33	13.33	10.00	10.00	10.00	15.00	0.66	1.00	1.50

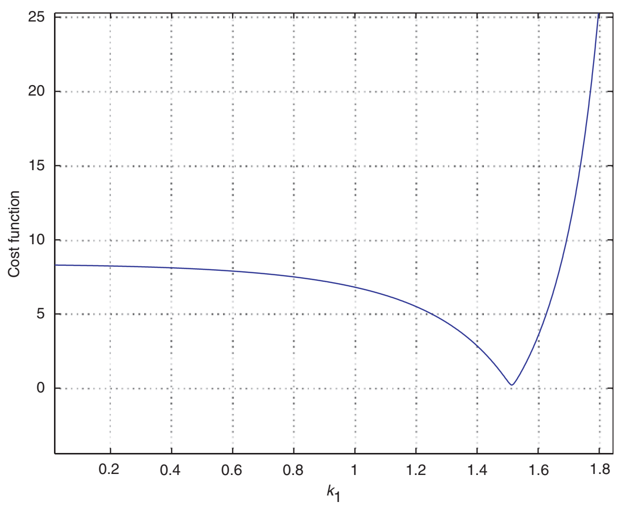

Figure 6. Cost function in covariance- structured approximation (method 2).

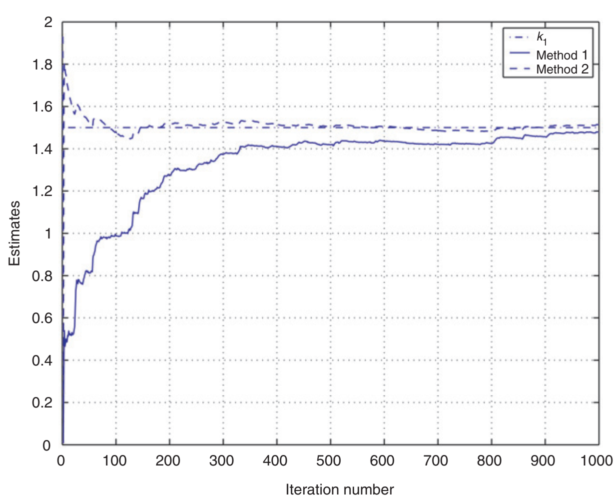

Figure 7. Estimates of k_1 using methods 1 and 2 as functions of data length.

is minimised is shown in Figure 6. The minimum was found to be insensitive to the choice of norm (Frobenious or maximum singular value) and weighting function W.

The main advantage of using the (computationally more demanding) covariance structured approximation method (method 2) for estimating k_1 is illustrated in Figure 7. This shows how the estimates of k_1 for the two schemes vary with the length of the data records (note that only method 1 is truly recursive). It can be seen from Figure 7 that the estimates based on method 2 converge much faster to the true parameter value $k_1 = 1.5$. This was consistently observed in all simulations and is not surprising as the full structure of the covariance matrix is used.

6.3. *Estimation method 3: inverse-covariance approximation*

The third estimation scheme is based on the observation that the inverse of the covariance matrix $P = \mathrm{Cov}(IP_2, Y_{2,1})$ defined in Equation (9) is quadratic in k_1^{-1}; thus estimating k_1 using an inverse-structured approximation (formulated in terms of the Frobenious norm) reduces to the solution of a scalar quartic equation.

Proposition 2: *Let $P = Cov(IP_2, Y_{2,1})$ be defined as in Equation (9), with $\sigma = 1$. Then, P^{-1} is quadratic in k_1^{-1}; in particular: $P^{-1} = A(k_2)\hat{k}_1^2 + B(k_2)\hat{k}_1 + C(k_2)$ where:*

$$A(k_2) = \begin{pmatrix} k_2^3(2-k_2) & k_2^2(2-k_2) \\ k_2^2(2-k_2) & k_2(2-k_2) \end{pmatrix}$$

$$B(k_2) = \begin{pmatrix} -2k_2^2(2-k_2)(k_2-1) & k_2(2-k_2)(1-2k_2) \\ k_2(2-k_2)(1-2k_2) & (k_2-1)(2-k_2) \end{pmatrix}$$

and

$$C(k_2) = \begin{pmatrix} k_2(2-k_2)(k_2-1)^2 & k_2(2-k_2)(k_2-1) \\ k_2(2-k_2)(k_2-1) & -(k_2-1)^2 \end{pmatrix}$$

in which $\hat{k}_1 = 1/k_1$.

Proof: Follows by direct calculations. Details are omitted. □

Consider now the inverse-covariance matrix approximation problem:

$$\min_{k_1 \in (0,2]} \|\hat{P} - \tilde{P}\|_F^2$$

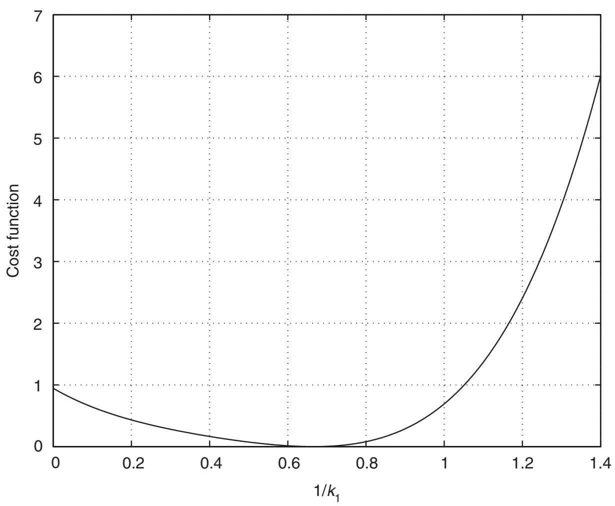

Figure 8. Cost function in inverse-covariance estimation.

where $\hat{P}$ denotes the empirical inverse-covariance matrix corresponding to $\tilde{P}$ (constructed from the data). The following result shows that this minimisation is equivalent to the minimisation of a fourth-order polynomial:

Proposition 3: *The optimisation*:

$$\gamma_\circ = \min_{\hat{k}_1 \in [1/2,\infty)} \|\tilde{P} - \hat{P}\|_F^2$$

is equivalent to:

$$\gamma_\circ = \min_{\hat{k}_1 \in [1/2,\infty)} \left\{\alpha_0 + \alpha_1 \hat{k}_1 + \alpha_2 \hat{k}_1^2 + \alpha_3 \hat{k}_1^3 + \alpha_4 \hat{k}_1^4\right\}.$$

where:

$$\alpha_0 = \mathrm{trace}(DD'), \quad \alpha_1 = -\mathrm{trace}(DB' + BD'),$$
$$\alpha_2 = \mathrm{trace}(BB' - DA' - AD'), \quad \alpha_3 = \mathrm{trace}(BA' + AB')$$
$$\alpha_4 = \mathrm{trace}(AA')$$

in which $D = C - \hat{P}$ *and* $A = A(k_2)$, $B = B(k_2)$ *and* $C = C(k_2)$ *are defined in Proposition* 2.

Proof: See Appendix 3, where detailed expressions of the α_i's are also included. □

Next we consider the polynomial:

$$p(\hat{k}_1) = \alpha_0 + \alpha_1 \hat{k}_1 + \alpha_2 \hat{k}_1^2 + \alpha_3 \hat{k}_1^3 + \alpha_4 \hat{k}_1^4.$$

The minimisation problem now involves simply the computation of the first derivative $p'(\hat{k}_1)$, which must be set equal to 0. Now,

$$p'(\hat{k}_1) = \alpha_1 + 2\alpha_2 \hat{k}_1 + 3\alpha_3 \hat{k}_1^2 + 4\alpha_4 \hat{k}_1^3 = 0 \qquad (10)$$

The solution of the above equation gives three roots from which we need to identify the real root in the interval $(1/2) \leq (1/k_1) < \infty$ corresponding to the minimising solution (we may also need to compare with the value of the function at $k_1 = (1/2)$ in case the minimum occurs at the edge of the allowable interval). The optimal estimate k_1 is then obtained by inverting the minimising $\hat{k}_1$.

Example: We demonstrate the inverse covariance-estimation method with the aid of a simulation example. We consider the three-node supply-chain with the same parameters used in previous estimation examples. We set $k_1 = k_2 = 1.5$, $SP_1 = SP_2 = 20$, $IP_1(0) = IP_2(0) = 10$ and have run the simulation for $n = 1000$ time steps.

The solution of the cubic equation gives three roots; two of these are complex $r_1 = 0.2855 + 0.3884i$ and $r_2 = 0.2855 - 0.3884i$ and one real $r_3 = 0.6657$, which corresponds to the minimum value $\hat{k}_1 = (1/k_1)$ of Equation (10). The estimated parameter is $\hat{k}_1 = 1.5021$, a value which is very close to the true parameter $k_1 = 1.5$. Figure 8 illustrates the minimised cost function in this case.

Once an accurate estimate of k_1 has been obtained, node 2 can switch to the optimal policy k_2^* and IP_2^*, thus minimising its average inventory level and its inventory fluctuations. To assess the 'value of information' when policy parameter k_1 is disclosed to the Distributor, it is necessary to carry out a full analysis of the estimation schemes presented above in order to determine the statistical properties of the estimate (e.g. variance, confidence intervals, rate of convergence, etc) and their dependance on data lengths. This analysis is not undertaken here and will be addressed in future work.

7. Conclusion

A novel state-space model has been presented for analysing the effect of proportional policies in multi-node supply-chains. Effective computational schemes were developed for calculating the covariance matrix of the model in closed-form (i.e. as a function of the policy parameters) using symbolic computations, including a recursive scheme on the number of nodes. This allows for a full characterisation of the bullwhip effect in three-node models, and leads to the formulation and solution of a constrained inventory minimisation problem under information-sharing. It has been shown that under information-sharing, selfish policies cannot lead to demand amplification. Finally, local estimation schemes have been investigated in the absence of information-sharing.

Although the main purpose of our work was to analyse a generic supply-chain model under continuous proportional policies, we believe that our main results contribute in the understanding of complex dynamic phenomena in real supply-chains (such as the bullwhip effect) and reinforce some of the main conclusions drawn by other researches based on alternative policies. Thus, volatile customer demand profiles combined with aggressive ordering policies have the tendency to reinforce the bullwhip effect, especially in multi-node systems. This suggests that forecasting policies that 'smooth-out' rapid short-term random fluctuations can have an important effect in alleviating the bullwhip effect (in this connection the intuition drawn from classical control engineering, especially frequency-domain methods, can be particularly beneficial (Dejonckheere et al. 2003)). In addition, enforcing co-operation between supply-chain participants has also been identified by many researches as an effective tool for suppressing amplification phenomena, a conclusion, which is supported by the results of Section 5 of this work. Many methods have been proposed for 'engineering' co-operation in supply-chains, including the undertaking of contractual obligations between participants. In general, modelling co-operation in supply-chains is highly complex, as participants are potential competitors and also need to co-operate to some extent to ensure the smooth operation of the chain. In Section 5, it was shown that subject to a probabilistic 'cooperation-type' constraint related to guaranteeing a minimum customer service level, 'selfish' policies (related to the minimisation of costs due to excessive inventory fluctuations) cannot give rise to demand amplification in the chain under information sharing between neighbouring nodes. In practise, many companies are reluctant to disclose customers' information, which they regard as proprietary. Thus, it is important to investigate whether policy parameters of adjacent nodes can be effectively estimated from (local) data in the absence of information-sharing, a topic that was briefly investigated in Section 6 of the article. Although the results obtained seem promising, further work is needed to determine whether sufficiently accurate estimates can be obtained with the proposed methods under realistic conditions (small data records, drift in customer demand parameters, etc).

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Appendix 1: Model augmentation

We consider the series supply-chain model with $n+1$ nodes depicted in Figure 1. Each node i ($i = 1, 2, \ldots, n$) has two inputs $w_{i,l}(t)$ and $w_{i,r}(t)$ and two outputs $z_{i,l}(t)$ and $z_{i,r}(t)$ (left and right respectively). It can be inferred from the nature of the figure's interconnections that $w_{i,l}(t) = z_{i-1,1}(t)$ and $w_{i,r}(t) = z_{i+1,1}(t)$. Note that, in contrast to all other nodes, the terminal $(n+1)$ node Φ has a single input and output. For $i=1,2,\ldots,n$ the discrete-time state-space model of the ith node's equations can be written as:

$$x_i(t+1) = A_i x_i(t) + \begin{pmatrix} B_{i,l} & B_{i,r} \end{pmatrix} \begin{pmatrix} w_{i,l}(t) \\ w_{i,r}(t) \end{pmatrix}$$

and

$$\begin{pmatrix} z_{i,l}(t) \\ z_{i,r}(t) \end{pmatrix} = \begin{pmatrix} C_{i,l} \\ C_{i,r} \end{pmatrix} x_i(t) + \begin{pmatrix} D_{i,ll} & D_{i,lr} \\ D_{i,rl} & D_{i,rr} \end{pmatrix} \begin{pmatrix} w_{i,l}(t) \\ w_{i,r}(t) \end{pmatrix}$$

We will assume that $D_{i,ll}=0$ for all $i=1,2,\ldots,n+1$ and $D_{i,lr}=D_{i,rl}=0$ for all $i=1,2,\ldots,n$ (we also define $D_{0,rr}=0$). These relations actually hold for the concrete supply-chain model, which is presented in Section 2. The equivalent state-space model of the $(n+1)$th-node is:

$$x_\phi(t+1) = A_\phi x_\phi(t) + B_\phi z_{n,r}(t) \text{ and } w_{n,r}(t) = C_\phi x_\phi(t)$$

where $x_\phi(t)$ denotes its state. Considering, for instance, a four-node model, the state-space equations as:

$$\begin{aligned} x_1(t+1) &= A_1 x_i(t) + B_{1,r} C_{2,l} x_2(t) + B_{1,l} w_{1,l}(t) \\ x_2(t+1) &= B_{2,l} C_{1,r} x_1(t) + (A_2 + B_{2,l} D_{1,rr} C_{2,l}) x_2(t) \\ &\quad + B_{2,r} C_{3,l} x_3(t) \\ x_3(t+1) &= B_{3,l} C_{2,r} x_2(t) + (A_3 + B_{3,l} D_{2,rr} C_{3,l}) x_3(t) \\ x_\phi(t+1) &= C_{3,r} x_3(t) + D_{3,rr} x_\phi(t). \end{aligned}$$

which can be assembled in matrix form to give the overall model of the chain.

The general $(n+1)$th-node model shown in Figure 1 can be aggregated as:

$$x(t+1) = \Psi x(t) + \Gamma w_{1,l}, \quad x(t) = [x_1'(t)\ x_2'(t)\ \ldots\ x_n'(t)\ x_\phi'(t)]'$$

where,

$$\Psi = \begin{pmatrix} \Psi_{11} & \Psi_{12} & 0 & \cdots & 0 \\ \Psi_{21} & \Psi_{22} & \Psi_{23} & \ddots & \vdots \\ 0 & \Psi_{32} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \Psi_{nn} & \Psi_{n\,n+1} \\ 0 & \cdots & 0 & \Psi_{n+1\,n} & \Psi_{n+1\,n+1} \end{pmatrix} \qquad \Gamma = \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_n \\ \Gamma_{n+1} \end{pmatrix}$$

More specifically, the elements of Ψ and Γ are defined as:

$$\Psi_{ij} = \begin{cases} A_i + B_{i,l} D_{i-1,rr} C_{i,l} & \text{for } i=j;\ i=1,2,\ldots,n+1 \\ B_{i,r} C_{i+1,l} & \text{for } i=j-1;\ i=1,2,\ldots,n \\ B_{i,l} C_{i-1,r} & \text{for } i=j+1;\ i=2,3,\ldots,n+1 \\ 0 & \text{for } |i-j|>1. \end{cases}$$

and $\Gamma_i = B_{i,l}$, if $i=1$; $\Gamma_i=0$ otherwise.

Appendix 2: Recursive implementation of k_1 and σ

The parameters k_1 and σ can be estimated recursively (Section 6.1) by the following scheme (where n is the iteration index):

$$\begin{aligned} \hat{P}_{22}(n) &= \frac{\hat{P}_{22}(n-1)}{1 + (Y_{2,1}(n) - \overline{Y}_{2,1}(n))^2 \hat{P}_{22}(n-1)} \\ \hat{\alpha}(n) &= \hat{\alpha}(n-1) + \hat{P}_{22}(n)(Y_{2,1}(n) - \overline{Y}_{1,2}(n)) \\ &\qquad [(IP_2(n) - \overline{IP}_2(n)) - (Y_{2,1}(n) - \overline{Y}_{2,1}(n))\hat{\alpha}(n-1)] \\ \hat{k}_1(n) &= \frac{1 + \hat{\alpha}(n) k_2}{1 + \hat{\alpha}(n)(k_2 - 1)} \\ \hat{\sigma}^2(n) &= \frac{\hat{P}_{22}(n)(2 - \hat{k}_1(n))}{\hat{k}_1(n)} \end{aligned}$$

where $\overline{IP}_2(n)$ and $\overline{Y}_{2,1}(n)$ denote running estimates of the means of IP_2 and $Y_{2,1}$, respectively. The recursion can be initialised from arbitrary initial conditions $\hat{P}_{22}(0) > 0$ and $\hat{\alpha}(0)$.

Appendix 3: Proof of Proposition 3

The Frobenius norm $\|\tilde{P} - \hat{P}\|_F^2$ can be written as:

$$\begin{aligned} \|\hat{P} - \tilde{P}\|_F^2 &= \text{trace}\{(\hat{P} - \tilde{P})(\hat{P} - \tilde{P})'\} \\ &= \text{trace}\{(D - B\hat{k}_1 - A\hat{k}_1^2)(D' - B'\hat{k}_1 - A'\hat{k}_1^2)\} \\ &= \text{trace}(DD') - \hat{k}_1 \text{trace}(DB' + BD') \\ &\quad + \hat{k}_1^2 \text{trace}(BB' - DA' - AD') \\ &\quad + \hat{k}_1^3 \text{trace}(BA' + AB') + \hat{k}_1^2 \text{trace}(AA') \\ &= \alpha_0 + \alpha_1 \hat{k}_1 + \alpha_2 \hat{k}_1^2 + \alpha_3 \hat{k}_1^3 + \alpha_4 \hat{k}_1^4. \end{aligned}$$

The proof is concluded on noting that as k_1 varies over $(0, 2]$, $\hat{k}_1$ varies over $[(1/2), \infty)$.

The coefficients α_i defined above may be written in closed form as:

$$\begin{aligned} \alpha_0 &= [\hat{P}_{11} - k_2(2-k_2)(k_2-1)^2]^2 \\ &\quad + 2[\hat{P}_{12} - k_2(2-k_2)(k_2-1)]^2 + [\hat{P}_{22} + (k_2-1)^2]^2 \\ \alpha_1 &= 4[\hat{P}_{11} - k_2(2-k_2)(k_2-1)^2] k_2^2 (2-k_2)(k_2-1) \\ &\quad + 4[\hat{P}_{12} - k_2(2-k_2)(k_2-1)] k_2 (2-k_2)(2k_2-1) \\ &\quad - 4[\hat{P}_{22} + (k_2-1)^2](k_2-1)^2 \\ \alpha_2 &= 4k_2^2(2-k_2)^2(k_2-1)^2 + 2k_2^2(2-k_2)^2(2k_2-1)2 \\ &\quad - 2[\hat{P}_{11} - k_2(2-k_2)(k_2-1)^2] k_2^3 (2-k_1) \\ &\quad - 4[\hat{P}_{12} - k_2(2-k_2)(k_2-1)] k_2^2 (2-k_2) \\ &\quad + 4(k_2-1)^4 - 2[\hat{P}_{22} - (k_2-1)^2] k_2 (2-k_2) \\ \alpha_3 &= -4k_2^5(2-k_2)^2(k_2-1) - 4k_2^3(2-k_2)^2(2k_2-1) \\ &\quad + 4k_2(2-k_2)(k_2-1)^2 \\ \alpha_4 &= k_2^6(2-k_2)^2 - 2k_2^4(2-k_2)^2 + k_2^2(2-k_2)^2 \end{aligned}$$

□